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MHD flow in an annular channel; theory and experiment

By J. A. BAYLIS
Central Electricity Research Laboratories,
Leatherhead, Surrey

AND J. C. R. HUNT
Department of Applied Mathematics and Theoretical Physics, Cambridge

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The theory of Hunt & Stewartson (1965) for MHD flow in a rectangular duct with conducting walls parallel and non-conducting walls perpendicular to the magnetic field is applied to the problem of electrically driven MHD flow in a rectangular annulus. It is assumed that the Hartmann number M is sufficiently great for secondary flow effects to be negligible. The experiment described here satisfied the conditions of the theory and thus provides a sensitive experimental check on Hunt & Stewartson's theory. The theory is found to agree with the experiments to within the accuracy of the asymptotic theory.

1. Introduction

There have not been many MHD experiments to measure laminar flow in ducts, particularly where the walls are electrically conducting. Such walls are to be found in most practical devices involving MHD flows in order that electric currents can pass between the walls and the fluid. An idealized MHD generator/pump of this kind was analyzed by Hunt & Stewartson (1965), (H & S), the main feature of the theory being the calculation of the flow in the boundary layers on the electrodes. Thus the theory can only properly be tested by an experimental flow in which the effects of the boundary layer contribute significantly to the total flow rate. In the experiments described here, where a reasonable comparison is possible, the flow was driven round an annulus of square cross-section by an imposed radial current perpendicular to an axial magnetic field. On account of the curvature of the duct some alterations are necessary to the theory of H & S, which are given in §2.

For recent reviews of other experiments where comparison with theory has been made, see Branover & Tsinober (1970) or Hunt & Shercliff (1971).

2. Theory

Consider the flow in a rectangular annulus shown in figure 1. The radii of the two concentric walls are r_1, r_2 and the height is $2a$, the walls parallel and perpendicular to the field being perfectly conducting and non-conducting respectively. The fluid is incompressible with conductivity σ and viscosity η. A current I flowing from one electrode to the other drives the fluid. If we non-dimensionalize in terms of I, the equations for the velocity v_θ, the induced magnetic field h_θ, and the potential ϕ, following Hunt & Williams (1968), are

$$0 = M\,\partial h/\partial\zeta + D_\rho(\rho v) + \partial^2 v/\partial\zeta^2, \tag{2.1}$$

$$0 = M\,\partial v/\partial\zeta + D_\rho(\rho h) + \partial^2 h/\partial\zeta^2, \tag{2.2}$$

$$-\partial h/\partial\zeta = -\partial\Phi/\partial\rho + Mv, \quad (1/\rho)\,\partial(\rho h)/\partial\rho = -\partial\Phi/\partial\zeta, \tag{2.3}$$

where $$D_\rho = \partial/\partial\rho(1/\rho\;\partial/\partial\rho), \quad v = \frac{v_\theta}{I/(4\pi a(\sigma\eta)^{\frac{1}{2}})}, \quad h = \frac{h_\theta}{I/(4\pi a)},$$

$$\Phi = \frac{\phi}{I/(2\pi a\sigma)}, \quad \rho = r/a, \quad \zeta = z/a, \quad \text{and} \quad M = B_0 a(\sigma/\eta)^{\frac{1}{2}}.$$

Also let $$R = \tfrac{1}{2}(r_1 + r_2).$$

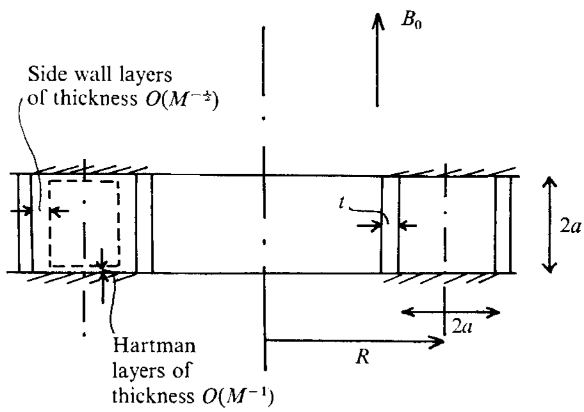

FIGURE 1. Sketch of an annulus with an axial magnetic field, showing the boundary layers which exist when $M \gg 1$.

In these equations we have assumed that radial and axial velocities are negligible, so that the only components of velocity are azimuthal and the only components of current are radial and axial. To find how large M must be for this assumption to be valid consider the equation for azimuthal *vorticity* in the Hartmann boundary layer. Since $v_r \gg v_z$ in this region we find

$$-\tilde{\rho}\,\partial(v_\theta^2/r)/\partial z = -\sigma B_0^2\,\partial v_r/\partial z + \eta\;\partial^3 v_r/\partial z^3,$$

whence it follows that

$$v_r/v_\theta = O\left[\frac{a}{R} \times \frac{\tilde{\rho} v_c}{\sigma B_0^2 a}\right],$$

where $\tilde{\rho}$ is the fluid density and v_{θ_c} is the core velocity. Thence we can estimate the size of the inertial terms which we have neglected from (2.1). A typical term is

$$\tilde{\rho} v_\theta\, v_r/r = O\left[\frac{\tilde{\rho}^2 v_{\theta c}^3}{\sigma B_0^2 R^2}\right],$$

whereas the viscous term on the right-hand side is

$$\eta\, \partial^2 v_\theta / \partial z^2 = O[\eta v_c\, M^2/a^2].$$

Thus the inertial term is negligible if

$$\frac{a^2}{R^2} \times \frac{Re^2}{M^4} \ll 1 \quad \text{or} \quad \left(\frac{K}{M^2}\right)^2 \frac{a}{R} \ll 1, \tag{2.4}$$

where $K = (2a/R)^{\frac{1}{2}} Re$ and $Re = 2av_{\theta_c}\tilde{\rho}/\eta$. This condition is less stringent than $(K/M^2)^2 \ll 1$ deduced by Baylis (1966), which was not correct. A slightly more stringent condition, $(K/M^2)^2\, M^{\frac{1}{2}} \ll 1$, must be satisfied if secondary flow effects are to be negligible in the boundary layers at $\rho = \rho_1, \rho_2$.

Boundary conditions

On both the walls at $\zeta = \pm 1$, and $\rho = \rho_1, \rho_2$, the condition on the current is that

$$j_z = \partial(\rho h)/\partial\rho = 0. \tag{2.5}$$

Thence the conditions on v and h become

$$\left.\begin{array}{lll} \text{at} & \zeta = \pm 1; & v = 0, \quad h = \pm 1/\rho, \\ \text{at} & \rho = \rho_1, \rho_2; & v = 0, \quad \partial(\rho h)/\partial\rho = 0. \end{array}\right\} \tag{2.6}$$

Solution

We use the method of H & S in which the flow is divided up into regions. By adding and subtracting (2.1) and (2.2) we obtain the combined equations in v and h,

$$O = \frac{M}{\rho}\frac{\partial}{\partial\zeta}[\rho(v \pm h)] \pm \left(D_\rho + \frac{1}{\rho}\frac{\partial^2}{\partial\zeta^2}\right)[\rho(v \pm h)]. \tag{2.7}$$

Away from the side walls at $\rho = \rho_1, \rho_2$, we need only consider the core flow and the Hartmann (or primary) boundary layers. Since $v \pm h$ is constant across the layer at $\zeta = 1$, and $v - h$ is constant at $\zeta = -1$, it follows that in the core

$$v + h = 1/\rho, \quad v - h = 1/\rho,$$

where

$$v = v_c = 1/\rho, \quad h = h_c = 0. \tag{2.8}$$

Thus the velocity in the core decreases radially outwards, while all the current is contained in the Hartmann layers of thickness $O(M^{-1})$, as shown in figure 5 (*b*) of H & S.

Now consider the boundary layers on the side walls, referred to as region (*c*) by H & S. The thickness of these layers is $O(M^{-\frac{1}{2}})$. Taking the wall at $\rho = \rho_1$, we write

$$v = v_c - Mv_s/\rho_1, \quad h = h_c + Mh_s/\rho_1.$$

Then if $X = (v_s + h_s)$, X has to satisfy

$$\partial^2 X/\partial\rho^2 + M\,\partial X/\partial\zeta = -(1/\rho)\,\partial X/\partial\rho + X/\rho^2 - \partial^2 X/\partial\zeta^2. \tag{2.9}$$

Now in (*c*) $\partial/\partial\rho = O(M^{\frac{1}{2}})$, so that if we put the right-hand side of (2.9) equal to zero we are ignoring terms of $O(aM^{\frac{1}{2}}/R), O(a^2/R^2)$ and $O(1)$ compared with terms of order M. Then, to the same order, the condition (2.6) on h becomes

$$\partial h_s/\partial\rho = 0 \quad \text{on} \quad \rho = \rho_1.$$

Using the symmetry of v and h it follows that the boundary conditions on X are

$$\left.\begin{array}{l} X(\zeta)+X(-\zeta) = 1/M \\ \partial/\partial\rho[X(\zeta)-X(-\zeta)] = 0 \end{array}\right\} \quad \text{at} \quad \rho = \rho_1, \tag{2.10}$$

and, because of the inner Hartmann layer, region (d), at

$$\zeta = 1, \quad X = 0. \tag{2.11}$$

The equation (2.9) with zero right-hand side together with (2.10) and (2.11) constitutes the same problem as that solved by H & S. Using their result it follows that

$$\int_0^\infty \int_{-1}^1 v_s \partial\bar{\rho}\,\partial\xi = \frac{(-\frac{1}{4})!\,2^{\frac{1}{2}}}{(\frac{1}{4})!\,M^{\frac{3}{2}}} \quad \text{where} \quad \bar{\rho} = (\rho-\rho_1)\,M^{\frac{1}{2}}. \tag{2.12}$$

For comparison with experiment we are interested in the overall relation between I and

$$Q = \int_{r_1}^{r_2}\int_{-a}^{a} v_\theta\, dr\, dz.$$

Integrating the core flow, (2.8), and using (2.12) we find

$$Q = \frac{-aI\ln(r_2/r_1)}{2\pi(\sigma\eta)^{\frac{1}{2}}}\left[1-\frac{0{\cdot}956(1/r_2+1/r_1)a}{M^{\frac{1}{2}}\ln(r_2/r_1)}-\left\{\frac{1}{M}+O\left(\frac{a^2(r_2^2-r_1^2)}{r_1^2r_2^2}M^{-1}\right)\right\}\right],$$

the undetermined term of $O(M^{-1})$ being produced by the neglected term on the right-hand side of (2.9). Q cannot easily be measured directly whereas the fall in potential, $\Delta\phi$, between the two walls at $r = r_1, r_2$ can. To find Q we integrate (2.3) across the duct

$$\Delta\phi = B_0 Q/2a - I\ln(r_2/r_1)/(4\pi a\sigma). \tag{2.13}$$

3. Experimental results

The experimental apparatus is described in detail by Baylis (1966), but there are two aspects of the apparatus that must be considered when comparing the experimental results with the theory. The first is that the theory assumes no contact resistance between the fluid and the conducting walls, but experimentally some contact resistance does exist between copper and mercury. There seems to be disagreement how important this is in MHD experiments. Glaberson, Donnelly & Roberts (1968) suggested that it was too large to permit any accurate duct flow experiments with conducting walls, whereas we found that with a good amalgam layer on the copper this resistance is of the order of 10^{-10} ohm m^2 and is negligible compared with the resistance of the flow when $M \gg 1$. A similar order of contact resistance between mercury and copper was also found by Hunt & Malcolm (1968) and Alty (1966). The second point concerns the resistance of the walls, for unless the conductance of the side walls is very much greater than that of the side wall boundary layers, i.e.

$$M^{\frac{1}{2}}\sigma_w t/(\sigma a) \gg 1, \tag{3.1}$$

where σ_w and t are the conductivity and thickness of the walls, our assumption that these walls are highly conducting is wrong. This condition (3.1) was well satisfied for $M \gg 16$ when

$$M^{\frac{1}{2}}\sigma_w t/(\sigma a) > 200.$$

In analyzing the flow in the secondary flow régime Baylis (1971) uses the variable

$$F = fRe = 2IB_0a^2/[\pi(R/a)\eta Q].$$

In looking at the results when there is no secondary flow it is more convenient to tabulate $P = F/2M$ as a function of R/a and M. Then from our theory, for a square annular duct

$$P = 1\bigg/\left[\frac{R}{2a}\ln\left(\frac{R+a}{R-a}\right)\left\{1-\frac{0{\cdot}956(2Ra)}{M^{\frac{1}{2}}(R^2-a^2)\ln[(R+a)/(R-a)]}-M^{-1}+O((a/R)M^{-1})\right\}\right].$$

In table 1 we compare the experimental values for P with the theoretical.

M	R/a	P_{exp}	P_{theory}
16·31	34	$1{\cdot}4 \pm 0{\cdot}1$	1·41
16·37	17	$1{\cdot}5 \pm 0{\cdot}1$	1·41
16·74	8	$1{\cdot}4 \pm 0{\cdot}1$	1·41
32·37	17	$1{\cdot}19 \pm 0{\cdot}04$	1·25
32·86	8	$1{\cdot}23 \pm 0{\cdot}04$	1·24
64·93	8	$1{\cdot}12 \pm 0{\cdot}03$	1·15
65·8	3·5	$1{\cdot}12 \pm 0{\cdot}03$	1·12
129·8	3·5	$1{\cdot}045 \pm 0{\cdot}02$	1·062

TABLE 1

P_{exp} was obtained by averaging the values of P in the range of M and K where P did not vary with K/M^2. See figure 3 of Baylis (1971). Allowance was made for the temperature variation of viscosity in the calculation of M and P. Another factor that had to be considered was the uncertainty in the duct dimension a caused by the growth of the amalgam layers on the copper surfaces. Typically these layers thickened by 25 μm during an experiment, which implies a maximum uncertainty in a of about 1·5 %.

4. Discussion

The comparison between the theoretical and the experimental results is good, the small systematic error (i.e. $P_{\text{theory}} > P_{\text{exp}}$) being of the order of the terms not calculated in the theoretical expression. The agreement provides the first experimental check on the theory of H & S for the side wall boundary layer. The check is a good one because the boundary-layer flux deficit is 25 % of the total flux in some of these cases. Note that the theory of H & S is only valid when $M \gg 1$, but in fact a comparison with the numerical calculations of Tani (1962) shows that the asymptotic theory is accurate to 2 % for M as low as 15. Thus the reasonable agreement with experiment at $M \sim 16$ should not be surprising.

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